

DEFORMATION OF EINSTEIN METRICS AND L^2 COHOMOLOGY ON STRICTLY PSEUDOCONVEX DOMAINS

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ABSTRACT. We construct new complete Einstein metrics on smoothly bounded strictly pseudoconvex domains of Stein manifolds. The approach that we take here is to deform the Kähler-Einstein metric constructed by Cheng and Yau—this is a generalization of the work of Biquard on the deformations of the complex hyperbolic metric on the unit ball. Recasting the problem into the question of vanishing of an L^2 cohomology and taking advantage of the asymptotic complex hyperbolicity of the Cheng–Yau metric, we establish the possibility of such a deformation when the dimension is at least three.

INTRODUCTION

Let Ω be a smoothly bounded strictly pseudoconvex domain of a Stein manifold of dimension $n \geq 2$. It is shown by Cheng and Yau [13] that then Ω carries a complete Kähler-Einstein metric g with negative scalar curvature, which is unique up to homothety. This metric has attracted much interest in connection with CR geometry of the boundary $\partial\Omega$. Actually, the induced CR structure on $\partial\Omega$, which is called the *conformal infinity* of g in our context, locally and asymptotically determines the metric g up to some high order, as first pointed out by Fefferman [20] and further investigated by Graham [23]. It is a version of bulk-boundary correspondence, which is more extensively studied in the setting of asymptotically *real* hyperbolic metrics (and in that of asymptotically anti-de Sitter metrics in physical context).

It was the idea of Biquard [5] that one can push forward the complex bulk-boundary correspondence toward asymptotically complex hyperbolic (ACH) Einstein metrics and CR structures that are not necessarily integrable. Those that are admitted as conformal infinities in the new setting are called partially integrable CR structures (or partially integrable almost CR structures) in the literature (see [9, 10]).

What Biquard showed in [5] was the perturbative existence and uniqueness result on the ball: for any partially integrable CR structure J on the sphere S^{2n-1} sufficiently “close” to the standard one, there exists an Einstein ACH metric on the unit ball $B^n \subset \mathbb{C}^n$ “close” to the complex hyperbolic metric, which is “locally unique” up to diffeomorphism action, whose conformal infinity is J . This result is parallel to that of Graham and Lee [24] for asymptotically real hyperbolic (AH) Einstein metrics.

In this paper, we take up the same perturbation problem on an *arbitrary* bounded strictly pseudoconvex domain Ω . Our seed metric is the Cheng–Yau metric, and by the *standard* CR structure on $\partial\Omega$ we shall mean the one induced by the complex structure of the ambient Stein manifold. Then we can establish the following generalization of Biquard’s result provided the dimension n is at least three.

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Theorem 0.1. *Let Ω be a smoothly bounded strictly pseudoconvex domain of a Stein manifold of dimension $n \geq 3$. Suppose that J is a partially integrable CR structure on the boundary $\partial\Omega$ sufficiently close to the standard CR structure in the $C^{2,\alpha}$ topology. Then there exists an Einstein ACH metric g on Ω with conformal infinity J . The metric g is locally unique modulo the action of diffeomorphisms on Ω inducing the identity on $\partial\Omega$.*

Here we are implicitly assuming that the underlying contact structure of J remains to be the standard one, so that the $C^{2,\alpha}$ -closeness between J and the standard CR structure is well-defined. As contact structures on closed manifolds are rigid (see Gray [25]), no generality is lost by this assumption.

The uniqueness is precisely stated as follows using the weighted Hölder space (see Subsection 1.2 for details): if g is our metric and \hat{g} is another Einstein ACH metric such that $\hat{g} - g \in C_\delta^{2,\alpha}(S^2T^*\Omega)$ for some $\delta > 0$ (which in particular implies that their conformal infinities are the same), then there exists a diffeomorphism $\Phi \in \text{Diff}(\Omega)$ that is at least continuous up to the boundary and restricts to the identity on $\partial\Omega$ for which $\Phi^*\hat{g} = g$. This uniqueness part is not a new result; it is discussed in [5, Proposition I.4.6].

Our proof of the existence is based on Biquard's framework. Namely, the proof is done by applying the inverse function theorem to a mapping between Banach spaces that is exactly the one considered in [5]. Then the problem becomes to see if the linearization of the mapping in question is isomorphic. This is relatively easy for the complex hyperbolic metric on the unit ball; however, for general domains, it is far less obvious. Here lies the main issue of our discussion.

Let us get into some more details. We consider the following Bianchi-gauged Einstein equation in order to get rid of the diffeomorphism invariance of the Einstein equation:

$$(0.1) \quad \mathcal{E}_g(h) := \text{Ric}(g+h) - \lambda(g+h) + \delta_{g+h}^* \mathcal{B}_g(h) = 0, \quad \mathcal{B}_g(h) := \delta_g h + \frac{1}{2} d \text{tr}_g h.$$

Here g denotes some ACH metric, h is a symmetric 2-tensor that is “small” compared to g (so that $g+h$ remains to be a metric in particular), δ_g is the divergence with respect to g , and δ_{g+h}^* is the formal adjoint of δ_{g+h} with respect to $g+h$. What we shall need is that the linearization \mathcal{E}'_g of the gauged Einstein operator (0.1) associated to the Cheng–Yau metric at $h=0$ is an isomorphism between certain weighted Hölder spaces of symmetric 2-tensors. An upshot of Biquard's theory (see also Lee [30] for the AH case) is that this isomorphism follows once the vanishing of the L^2 kernel of \mathcal{E}'_g is established. Therefore, the L^2 kernel, denoted by $\ker_{L^2} \mathcal{E}'_g$, is called the *obstruction space* of Einstein deformation.

The linearization \mathcal{E}'_g for an Einstein metric g turns out to be the following operator, where \mathring{R} denotes the usual action of the curvature tensor on symmetric 2-tensors:

$$(0.2) \quad \mathcal{E}'_g = \frac{1}{2}(\nabla^* \nabla - 2\mathring{R}).$$

This expression shows, in particular, that the L^2 kernel vanishes when g has negative sectional curvature everywhere (hence we get the result for the unit ball). The problem is that we cannot expect this level of knowledge for the sectional curvature of a general Cheng–Yau metric g_{CY} . However, the asymptotic curvature behavior of g_{CY} at infinity is known quite well: the holomorphic sectional curvature uniformly tends to a negative constant. This suggests that we can conquer the difficulty if it is reduced to an analysis *near infinity* in a suitable way.

Our crucial idea is to use Koiso's observation [29]. Note first that any symmetric 2-tensor σ can be decomposed into the sum of hermitian and anti-hermitian parts: $\sigma = \sigma_H + \sigma_A$. For a Kähler–Einstein metric g , (0.2) shows that this decomposition is respected by \mathcal{E}'_g . Now the duality induced by the metric identifies σ_A with a $(0,1)$ -form α with values in the holomorphic tangent

bundle $T^{1,0}$. Under this identification, it turns out that $\mathcal{E}'_g \sigma_A$ corresponds to $\frac{1}{2} \Delta \bar{\partial} \alpha$. Then, a little bit of further consideration leads to the conclusion that the vanishing of the obstruction space follows from

$$(0.3) \quad L^2 \mathcal{H}^{0,1}(\Omega; T^{1,0}) = 0,$$

where the left-hand side is the space of L^2 harmonic $T^{1,0}$ -valued $(0,1)$ -forms on Ω with respect to the Cheng–Yau metric.

By the de Rham–Hodge–Kodaira decomposition on noncompact manifolds, $L^2 \mathcal{H}^{0,1}(\Omega; T^{1,0})$ is isomorphic to the so-called *reduced* L^2 cohomology $L^2 H_{\text{red}}^{0,1}(\Omega; T^{1,0})$. Although it may not be the same as the usual L^2 cohomology $L^2 H^{0,1}(\Omega; T^{1,0})$, it is a trivial fact that if $L^2 H^{0,1}$ vanishes then so does $L^2 H_{\text{red}}^{0,1}$. Thus the vanishing of $L^2 \mathcal{H}^{0,1}$ reduces to that of the L^2 cohomology. A virtue of this reduction in terms of cohomology is that we have a certain long exact sequence (see Ohsawa [34]) that makes it sufficient to show that the L^2 cohomology *on a neighborhood of the boundary* vanishes. This problem is solved by applying a Bochner-type method based on the Morrey–Kohn–Hörmander equality, which was exploited in the classical $\bar{\partial}$ -Neumann problem, when $n \geq 4$.

For $n = 3$, we need an additional idea: we can further reduce the vanishing of $\ker_{L^2} \mathcal{E}'_g$ to that of a *weighted* L^2 cohomology. This is made possible by using a result of Biquard [5] again, which states that the elements of $\ker_{L^2} \mathcal{E}'_g$ on a general ACH-Einstein manifold actually satisfy a stronger decay property at infinity than just being L^2 ; in our context, in which we are aiming for the vanishing result, we might be able to say to express this fact that these elements have an *a priori* decay. Then the introduction of the weight improves our estimate enough to show the desired vanishing result. Unfortunately, the case $n = 2$ cannot be settled even if we use this additional technique; this case remains unsolved so far.

The argument outlined above is presented in detail in the following way. In Section 1, we summarize the definition of ACH metrics and the associated weighted Hölder spaces. After that Biquard’s Einstein deformation theory is recalled, and the *a priori* decay for the elements of $\ker_{L^2} \mathcal{E}'_g$ is explained. In Section 2, we review the L^2 cohomology on noncompact Hermitian manifolds, and the problem is reduced to the vanishing of the L^2 cohomology near the boundary. In Section 3, we establish the necessary estimate and complete the proof of Theorem 0.1.

We include an appendix, in which we give yet another proof of a result of Donnelly and Fefferman [16] on L^2 harmonic (scalar-valued) differential forms using the *a priori* decay technique. In fact, via this technique, the vanishing part of the Donnelly–Fefferman theorem boils down to the vanishing of some weighted L^2 cohomologies, which we can prove by the standard Bochner–Kodaira–Nakano formula.

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1. EINSTEIN DEFORMATION THEORY OF ASYMPTOTICALLY COMPLEX HYPERBOLIC METRICS

1.1. The Cheng–Yau metric. Cheng and Yau considered the Calabi problem on noncompact complex manifolds in [13]. In particular, they established the following theorem.

Theorem 1.1 (Cheng–Yau [13]). *Let Ω be a smoothly bounded strictly pseudoconvex domain of a Stein manifold of dimension $n \geq 2$. Then, for each fixed $\lambda < 0$, there exists a unique complete Kähler metric g on Ω satisfying $\text{Ric}(g) = \lambda g$.*

Such a metric is constructed by solving the complex Monge–Ampère equation. More specifically, normalizing the metric by setting $\lambda = -(n+1)$, we can obtain such an Einstein metric in the form $g = g_{i\bar{j}} dz^i d\bar{z}^j$ with

$$(1.1) \quad g_{i\bar{j}} = \frac{\partial^2(-\log \varphi)}{\partial z^i \partial \bar{z}^j} + h_{i\bar{j}},$$

where $\varphi \in C^\infty(\Omega) \cap C^{n+1,\alpha}(\bar{\Omega})$ is a positive defining function of Ω and $h_{i\bar{j}}$ is a hermitian symmetric form on the ambient Stein manifold Y ; when $Y = \mathbb{C}^n$ one can always take $h_{i\bar{j}} = 0$. In what follows, by the Cheng–Yau metric, this one is always referred to. The uniqueness of the metric [13, Theorem 8.3] follows from Yau’s Schwarz Lemma for volume forms [33, Section 1].

In the paper of Cheng and Yau, the existence and the boundary regularity of φ is stated explicitly only for domains of \mathbb{C}^n and of Kähler manifolds admitting metrics of negative Ricci curvature. To see that their result extends to the case of domains of Stein manifolds, take a positive defining function $\psi \in C^\infty(\bar{\Omega})$ so that $-\psi$ is strictly plurisubharmonic on $\bar{\Omega}$. Let g' be a Kähler metric on Y such that $g_{i\bar{j}} = \partial_i \partial_{\bar{j}}(-\log \psi) - r_{i\bar{j}} > 0$ on Ω , where $(n+1)r_{i\bar{j}}$ is the Ricci tensor of g' . Then by [13, Theorem 4.4], there exists a function $u \in C^\infty(\Omega)$ for which $g_{i\bar{j}} + u_{i\bar{j}}$ is a metric that is quasi-equivalent to $g_{i\bar{j}}$ and

$$\frac{\det(g_{i\bar{j}} + u_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{(n+1)u+F}$$

with

$$F = \log \left(\frac{\det(g'_{i\bar{j}})}{\psi^{n+1} \det(g_{i\bar{j}})} \right).$$

Then it can be seen that $g_{i\bar{j}} + u_{i\bar{j}}$ is an Einstein Kähler metric. The boundary regularity of $\varphi = e^{-u}\psi$ can be shown by slightly modifying [13, Section 6]; see also van Coevering [36, Subsection 3.4] discussing a similar case.

A direct computation shows that the holomorphic sectional curvature of the Cheng–Yau metric (1.1) uniformly tends to -2 at infinity (see [13, Equation (1.22)] for the case $Y = \mathbb{C}^n$). Namely,

$$(1.2) \quad R_{i\bar{j}k\bar{l}} = -(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) + o(1) \quad \text{at } \partial\Omega,$$

where the notation $o(1)$ means that this term has pointwise norm (with respect to g) that tends to zero uniformly at $\partial\Omega$. Moreover, the Cheng–Yau metric g has bounded geometry in the sense that its injectivity radius r_{inj} is positive and the curvature tensor, as well as its covariant derivative of arbitrary order, is bounded (cf. [11, 35]). One way to show that $r_{\text{inj}} > 0$ is applying a result of Ammann–Lauter–Nistor [1, Proposition 4.19]; actually, one can further show that the pointwise injectivity radius at $p \in \Omega$ tends to infinity as p approaches $\partial\Omega$. The boundedness of $\nabla^m R$, $m \geq 0$, can be seen by reducing it to the same property of the preliminary metric $\partial_i \partial_{\bar{j}}(-\log \psi) - r_{i\bar{j}}$, which follows by an explicit computation that makes use of the local frame employed in [32, Subsection 5.1]. The reduction is justified because $\nabla^m u$ is bounded for any

m as shown in the proof of [13, Theorem 4.4]. Alternatively, one can use the existence of an asymptotic expansion of φ involving logarithmic terms established by Lee–Melrose [31].

1.2. ACH metrics. The model of asymptotically complex hyperbolic metrics is given by the following metric on $M \times (0, 1)$, where x is the coordinate of the second factor:

$$(1.3) \quad g_0 = \frac{1}{2} \left(4 \frac{dx^2}{x^2} + \frac{\theta^2}{x^2} + \frac{L_\theta}{x} \right).$$

Here M is a closed strictly pseudoconvex partially integrable CR manifold (see, e.g., [32]). Recall that an almost CR manifold (M, H, J) is said to be *partially integrable* if

$$[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M \oplus \overline{T^{1,0}M}),$$

where $T^{1,0}M \subset \mathbb{C}H$ is the i -eigenbundle of $J \in \Gamma(\text{End}(H))$. The Levi form L_θ is defined as usual for any 1-form θ that exactly annihilates H , and so is the notion of strict pseudoconvexity of (M, H, J) . In formula (1.3), θ is always taken so that the Levi form L_θ is positive definite. We call such a θ a *pseudohermitian structure*.

Using the model metric (1.3), we define as follows (see also Biquard [5], Biquard–Mazzeo [6, 7]).

Definition 1.2. Let X be a noncompact smooth manifold of real dimension $2n$, where $n \geq 2$, which compactifies into a smooth manifold-with-boundary \overline{X} . Then a Riemannian metric g on X is called *asymptotically complex hyperbolic* (or ACH for short) if there exists a diffeomorphism between a neighborhood of $M := \partial\overline{X}$ in \overline{X} and $M \times [0, \varepsilon)$ under which there is some strictly pseudoconvex partially integrable CR structure (H, J) and a pseudohermitian structure θ on M such that

$$(1.4) \quad g = g_0 + k, \quad k \in C_\delta^{2,\alpha}(S^2T^*X)$$

for some $\delta > 0$, where g_0 is the model metric (1.3) for (M, H, J, θ) . The partially integrable CR structure (H, J) , or J , is called the *conformal infinity* of g .

Here the space

$$C_\delta^{2,\alpha}(S^2T^*X) := x^\delta C^{2,\alpha}(S^2T^*X)$$

is the weighted Hölder space of symmetric 2-tensors on X with respect to g_0 . In order to define the $C^{k,\alpha}$ Hölder norm of a section s of a tensor bundle E , we use the fact that there is a radius $r_{\text{conv}} > 0$, which is smaller or equal to r_{inj} , such that any two points $p, q \in X$ satisfying $d(p, q) < r_{\text{conv}}$ is connected by a unique minimizing geodesic [8]. Upon fixing a parameter $r < r_{\text{conv}}$, we define

$$\|s\|_{C^\alpha} := \sup|s| + \sup_{\text{dist}(p,q) < r} \frac{|\Pi_{p \rightarrow q}(s(p)) - s(q)|}{\text{dist}(p, q)^\alpha},$$

where $\Pi_{p \rightarrow q}: E_p \rightarrow E_q$ is the parallel transport along the minimizing geodesic from p to q , and

$$\|s\|_{C^{k,\alpha}} := \sum_{m=0}^{k-1} \sup|\nabla^m s| + \|\nabla^k s\|_{C^\alpha}.$$

For $s \in C_\delta^{k,\alpha}(E)$, we define

$$\|s\|_{C_\delta^{k,\alpha}} := \|x^{-\delta} s\|_{C^{k,\alpha}}.$$

Another choice of r gives an equivalent norm; hence the space $C_\delta^{k,\alpha}(E)$ remains unchanged.

Since g_0 has bounded geometry (as one can show in the same way as in the previous subsection), an equivalent Hölder norm can also be defined in terms of the trivialization with respect

to geodesic coordinate charts, for which there exists uniform constants $c > 0$ and $c_m > 0$ such that

$$(1.5) \quad c^{-1}\delta_{ij} < (g_0)_{ij} < c\delta_{ij}, \quad |\partial_{k_1} \dots \partial_{k_m}(g_0)_{ij}| < c_m$$

(here the indices correspond to real coordinates) by the result of Eichhorn [18]. One can further use any uniformly locally finite coordinate charts $\{(U_\lambda, \Phi_\lambda)\}$, where Φ_λ maps $U_\lambda \subset X$ onto the open ball $B_r(0) \subset \mathbb{R}^{2n}$ of radius $r > 0$ (independent of λ) centered at the origin, such that (1.5) holds and $\{\Phi_\lambda^{-1}(B_{r/2}(0))\}$ covers X . This implies in particular that a similar approach to that of “Möbius coordinates” taken by Lee [30] in the AH case can also be used; consequently, the Hölder spaces defined by any two different choices of (J, θ) are actually the same.

The Cheng–Yau metric g on a smoothly bounded strictly pseudoconvex domain Ω can be regarded as an ACH metric in the sense of Definition 1.2 via the “square root construction” of Epstein–Melrose–Mendoza [19] (see also [32, Example 2.4]). In this case, the compactification \overline{X} is topologically $\overline{\Omega}$, but the C^∞ -structure is replaced so that the square roots of boundary defining functions are regarded as smooth. Therefore, in this case,

$$C_\delta^{k,\alpha}(S^2 T^* X) = \varphi^{\delta/2} C^{k,\alpha}(S^2 T^* X),$$

where φ is any smooth positive defining function of Ω . Although the identity map $\iota: \overline{X} \rightarrow \overline{\Omega}$ does not have smooth inverse, its restriction to the interior X and to the boundary $\partial\overline{X}$ are diffeomorphisms onto Ω and $\partial\Omega$, respectively. Thus $\partial\overline{X}$ is identified with $\partial\Omega$. The conformal infinity of g is actually the standard CR structure on $\partial\Omega$, i.e., the one induced from the complex structure of the ambient Stein manifold.

1.3. Einstein deformations. Let g be an arbitrary ACH metric on X satisfying the Einstein equation, which is in fact forced to be

$$\text{Ric}(g) = -(n+1)g.$$

The conformal infinity of g , which is a partially integrable CR structure on $M = \partial\overline{X}$, is denoted by (H, J) .

Take a neighborhood of J in the set of all $C^{2,\alpha}$ partially integrable CR structures admitted by the contact distribution H . This is identified with a neighborhood of the origin in the Hölder space $C^{2,\alpha}(S^2(\wedge^{1,0}M))$ of anti-hermitian symmetric 2-tensors over $T^{1,0}M$. The identification is given as follows: for $\psi \in C^{2,\alpha}(S^2(\wedge^{1,0}M))$ with sufficiently small norm (with respect to the Levi form L_θ for J and some fixed pseudohermitian structure θ), expressed as $\psi_{\alpha\beta}$ with respect to a local frame $\{Z_\alpha\}$ of $T^{1,0}M$, the vector bundle

$$\psi T^{1,0}M = \text{span} \{ Z_\alpha + \psi_\alpha^{\bar{\beta}} \bar{Z}_{\bar{\beta}} \}$$

defines the corresponding $C^{2,\alpha}$ partially integrable CR structure J_ψ , where the second index of ψ is raised by the Levi form and Einstein’s summation convention is observed. We write $J_\psi = J + \psi$.

We first consider a preliminary extension of J_ψ to an ACH metric g_ψ . Identify a neighborhood of $\partial\overline{X}$ with $M \times [0, \varepsilon)$ so that g is as in (1.4). Fix a cutoff function $\chi \in C^\infty(\overline{X})$ that equals 1 near $\partial\overline{X}$ and is supported in $M \times [0, \varepsilon)$. Let $\tilde{\psi}$ be the difference of the Levi forms of J and J_ψ with respect to θ , and set

$$g_\psi = g + \frac{\chi \tilde{\psi}}{x^2}.$$

If $\|\psi\|_{C^{2,\alpha}}$ is sufficiently small, then g_ψ becomes an (in general non-smooth) ACH metric with conformal infinity J_ψ . Note that $g_0 = g$. The metric g_ψ is approximately Einstein in the sense that

$$\text{Ric}(g_\psi) + (n+1)g_\psi \in C_1^{0,\alpha}(S^2T^*X);$$

hence it is trivial that $\mathcal{E}_{g_\psi}(0) \in C_1^{0,\alpha}(S^2T^*X)$.

We define a map

$$(1.6) \quad Q: \mathcal{B} \rightarrow C^{2,\alpha}(S^2(\wedge^{1,0}M)) \oplus C_\delta^{0,\alpha}(S^2T^*X),$$

where $0 < \delta \leq 1$ and \mathcal{B} is a small neighborhood of $0 \in C^{2,\alpha}(S^2(\wedge^{1,0}M)) \oplus C_\delta^{2,\alpha}(S^2T^*X)$, by

$$(1.7) \quad Q(\psi, h) = (\psi, \mathcal{E}_{g_\psi}(h)).$$

We shall prove that this is bijective near the origin; then the inverse image of $(\psi, 0)$ contains only one element (ψ, h_ψ) close to the origin, for which the metric $g_\psi + h_\psi$ is an Einstein ACH metric with conformal infinity J_ψ by [5, Lemme I.1.4]. By the inverse function theorem, it suffices to show that the linearization of Q is an isomorphism between the Banach spaces. Since the first component of Q is just the identity map, what we have to verify is that

$$(1.8) \quad \mathcal{E}'_g: C_\delta^{2,\alpha}(S^2T^*X) \rightarrow C_\delta^{0,\alpha}(S^2T^*X)$$

is isomorphic.

The Fredholm theory of ACH metrics [5] shows that the operator (1.8) is an isomorphism if and only if the L^2 kernel vanishes. Here the L^2 kernel is the kernel of \mathcal{E}'_g understood as an unbounded operator with domain

$$\text{dom } \mathcal{E}'_g := \{ \alpha \in L^2(S^2T^*X) \mid \mathcal{E}'_g \alpha \in L^2(S^2T^*X) \},$$

which is called the maximal closed extension. Thus the space $\ker_{L^2} \mathcal{E}'_g$ is called the *obstruction space* of Einstein deformations, and when it vanishes g is called *nondegenerate*. We can summarize the discussion so far as in the next proposition (the uniqueness statement follows from [5, Proposition I.4.6]).

Proposition 1.3 (Biquard [5, Théorème I.4.8]). *Let g be a nondegenerate Einstein ACH metric with conformal infinity J . Then, for any $\psi \in C^{2,\alpha}(S^2(\wedge^{1,0}M))$ close enough to zero, there exists an Einstein ACH metric g' whose conformal infinity is $J_\psi = J + \psi$. The metric g' is locally unique in the sense that any Einstein ACH metric lying in a sufficiently small $C_\delta^{2,\alpha}$ -neighborhood of g' pulls back to g' by a diffeomorphism on X inducing the identity on $\partial\overline{X}$.*

In the rest of this subsection, we describe the fact that $\ker_{L^2} \mathcal{E}'_g = 0$ implies the isomorphicity of (1.8) in the context of general theory of geometrically defined elliptic linear differential operators for ACH metrics.

Let $P: \Gamma(E) \rightarrow \Gamma(F)$ be an elliptic differential operator of order m , where E and F are subbundles of $(TX)^{\otimes s} \otimes (T^*X)^{\otimes t}$ invariant under the action of $O(2n)$ (the group $O(2n)$ can be replaced by $U(n)$ for Kähler ACH metrics), with a universal expression in terms of the Levi-Civita covariant differentiation and the actions of the curvature tensor of g . Such an operator is called *geometric*. A consequence of this assumption is that P determines a well-defined mapping

$$C_\delta^{m,\alpha}(E) \rightarrow C_\delta^{0,\alpha}(E),$$

and also

$$H_\delta^m(E) \rightarrow L_\delta^2(E),$$

for an arbitrary $\delta \in \mathbb{R}$. Here $L_\delta^2(E)$ and $H_\delta^m(E)$ are weighted L^2 and L^2 -Sobolev spaces, which are defined by $L_\delta^2(E) := x^\delta L^2(E)$ and $H_\delta^k(E) := x^\delta H^k(E)$.

Another virtue of the geometricity is that it allows us to consider the operator P also on $\mathbb{C}H^n$. Then we can formulate the following coercivity assumption, which is crucial in the next lemma:

$$(1.9) \quad \|\alpha\|^2 \leq C\|P\alpha\|^2, \quad \alpha \in \text{dom } P \subset L^2(E), \quad \text{on } \mathbb{C}H^n.$$

Its validity for $P = \mathcal{E}'_g$ is obvious from (0.2).

We need in addition to introduce a nonnegative real number R_P called the *indicial radius* of P . For this, we again consider the operator P on $\mathbb{C}H^n$. Fixing an origin $o \in \mathbb{C}H^n$ identifies a subgroup $G = U(n)$ of the group of isomorphisms $G_0 = PSU(n, 1)$, which gives the expression

$$\mathbb{C}H^n \cong G_0/G.$$

Then E and F are expressed as the associated bundles $G_0 \times_G V$ and $G_0 \times_G W$, respectively, where V and W are representations of G . Furthermore, we fix a unit tangent vector $v_0 \in T_o \mathbb{C}H^n$ at the origin o , and let $\gamma(r) = \exp(rv_0)$ be the geodesic that it determines. Then the isotropy subgroup of the G -action on $T_o \mathbb{C}H^n$ gives a subgroup $H = U(n-1) \subset G$. To state it differently, H is the isotropy subgroup of the G -action on the sphere at infinity S^{2n-1} about the limit point of γ . Thus

$$S^{2n-1} \cong G/H.$$

We introduce an identification

$$\mathbb{C}H^n \setminus \{o\} \cong S^{2n-1} \times (0, \infty)$$

that maps $g \cdot \gamma(r)$ to the pair (gH, r) , where $g \in G$. Then, restricted on each concentric sphere $S_r := S^{2n-1} \times \{r\}$, the vector bundle E can be seen as a homogeneous bundle over G/H . If we identify the fiber $E_{\gamma(r)}$ with $V = E_o$ by the parallel transport along γ , then

$$E|_{S_r} \cong G \times_H V.$$

Therefore, a section of E (over $\mathbb{C}H^n \setminus \{o\}$) can be identified with a function on $G \times (0, \infty)$ with values in V that is H -equivariant. A similar identification can be introduced for sections of F .

We can write P down in terms of this expression of sections of E and F . Then it follows from [5, Equation (I.1.2)] that the derivatives tangent to G vanish at the limit $r \rightarrow \infty$. In fact, P has the following form, where each $a_i(r)$ is a function with values in the space $\text{Hom}_H(V, W)$ of H -equivariant linear mappings $V \rightarrow W$:

$$P = a_0(r)\partial_r^m + a_1(r)\partial_r^{m-1} + \cdots + a_m(r) + O(e^{-r}).$$

As $r \rightarrow \infty$, the coefficients $a_i(r)$ have well-defined limits $a_i \in \text{Hom}_H(V, W)$. A number $s \in \mathbb{C}$ is called an *indicial root* of P when

$$I_P(s) := (-1)^m a_0 s^m + (-1)^{m-1} a_1 s^{m-1} + \cdots + a_m \in \text{Hom}_H(V, W)$$

fails to be injective. Intuitively, the set Σ_P of indicial roots is such that any solution of $Pu = 0$ is expected to behave asymptotically like $u \sim u_0 e^{-sr}$ for some $s \in \Sigma_P$ and a section u_0 over S^{2n-1} .

For the formal adjoint operator $P^*: \Gamma(F) \rightarrow \Gamma(E)$, we can show that

$$I_{P^*}(s) = I_P(2n - \bar{s})^*.$$

The number $2n$ here should be understood as the twice the borderline weight of being in the L^2 space. (See also Lee [30, Proposition 4.4] for the AH case, in which the “weight of the tensor bundle” appears in the formula just because the value of the parameter s is shifted.) In

particular, when P is formally self-adjoint, then the indicial roots appear symmetrically about the line $\operatorname{Re} s = n$. In this case,

$$R_P := \min_{s \in \Sigma_P} |\operatorname{Re} s - n|$$

is called the *indicial radius* of P . Now we can formulate the following proposition.

Proposition 1.4 (Biquard [5, Proposition I.3.5]). *Assume that the operator P acting on sections of E is formally self-adjoint and satisfies the coercivity estimate (1.9) on $\mathbb{C}H^n$. Then, for $|\delta| < R_P$, the operator P seen as mappings*

$$(1.10) \quad C_{n+\delta}^{m,\alpha}(E) \rightarrow C_{n+\delta}^{0,\alpha}(E)$$

and

$$(1.11) \quad H_\delta^m(E) \rightarrow L_\delta^2(E)$$

are Fredholm with index zero, and the kernel of each of the mappings above equals $\ker_{L^2} P$.

For the linearized gauged Einstein operator \mathcal{E}'_g , the estimate (1.9) follows from (0.2). A computation of the indicial roots of \mathcal{E}'_g is given (or at least sketched) in [5, Section I.2.A and Lemme I.4.3], whose result is as follows. The G -representation associated to $E = S^2 T^* X$ is

$$V = S_{\mathbb{R}}^2 \mathfrak{m}_0^*,$$

where $\mathfrak{g}_0 = \mathfrak{g} \oplus \mathfrak{m}_0$ is the Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$ for the symmetric space $\mathbb{C}H^n = G_0/G$. The space \mathfrak{m}_0 is canonically identified with $T_o \mathbb{C}H^n$, and decomposes as $\mathbb{C}v_0 \oplus \mathbb{C}^{n-1}$ by the H -action. Since a_i 's are H -equivariant, by Schur's Lemma each H -irreducible component of $S_{\mathbb{R}}^2 \mathfrak{m}_0^*$ is mapped by the indicial polynomial $I_{\mathcal{E}'_g}(s)$ into the sum of isomorphic components. On $S_{\mathbb{C}}^2(\mathbb{C}^{n-1})^* \rightarrow S_{\mathbb{C}}^2(\mathbb{C}^{n-1})^*$, the indicial polynomial becomes $s^2 - 2ns$ times a nonzero constant, which gives indicial roots 0 and $2n$. The fact is that these are the closest roots to the borderline $\operatorname{Re} s = n$; hence we conclude that $R_{\mathcal{E}'_g} = n$.

The computation in [5] is described in terms of relevant Lie algebras. A more primitive, though probably less insightful, calculation is given in [32, Lemma 5.4]; the operator $\Delta_L + n + 2$ that appears in [32] is nothing but \mathcal{E}'_g up to a constant factor (note that ACH manifolds in [32] has real dimension $2n + 2$). Let us recall this computation for the spacial case of $\mathbb{C}H^n$, as it suffices for our purpose. Rather than using the unit ball model, we identify $\mathbb{C}H^n$ with $\mathcal{H}^{2n-1} \times (0, \infty)$, where \mathcal{H}^{2n-1} is the Heisenberg group, so that the complex hyperbolic metric is given by formula (1.3) with the standard pseudohermitian structure θ . Let T be the Reeb vector field on \mathcal{H}^{2n-1} and $\{Z_1, \dots, Z_{n-1}\}$ a local frame of the CR holomorphic tangent bundle. If we set $Z_\tau := \frac{1}{2}x\partial_x + ix^2T$ and $Z_\alpha := xZ_\alpha$, $\alpha = 1, \dots, n-1$, then $\{Z_\tau, Z_1, \dots, Z_{n-1}\}$ is a local frame of $T^{1,0}\mathbb{C}H^n$ and the Christoffel symbols are given by [32, Equations (5.2)]. We compute the action of \mathcal{E}'_g using this frame. Then we obtain, for example,

$$(\mathcal{E}'_g \sigma)_{\alpha\beta} = -\frac{1}{4}x\partial_x(x\partial_x - 2n)\sigma_{\alpha\beta} + O(x) \cdot (\text{derivatives of } \sigma \text{ in the } \mathcal{H}^{2n-1}\text{-direction}).$$

Although the hypersurfaces $\mathcal{H}_x = \mathcal{H}^{2n-1} \times \{x\}$ and the concentric spheres S_r in the unit ball model are different, if we identify the geodesic $\gamma(r)$ with the preimage of the origin $0 \in \mathcal{H}^{2n-1}$ by the first projection $\mathcal{H}^{2n-1} \times (0, \infty) \rightarrow \mathcal{H}^{2n-1}$, then the tangent spaces of \mathcal{H}_x and of S_r are the same along this geodesic. From this one can conclude that the indicial roots on the component $S_{\mathbb{C}}^2(\mathbb{C}^{n-1})^* \rightarrow S_{\mathbb{C}}^2(\mathbb{C}^{n-1})^*$ is as described in the previous paragraph. The other roots can also be read off from [32, Equations (5.9)].

Anyway, by Proposition 1.4, the mappings (1.10) and (1.11) for the linearized gauged Einstein operator are isomorphic for $|\delta| < n$ if the L^2 kernel vanishes. Proposition 1.3 follows by taking δ

close to $-n$. We remark, on the other hand, that our conclusion for $\delta > 0$ can be seen as giving an *a priori* decay of the elements of $\ker_{L^2} \mathcal{E}'_g$, which has applications in the proof of the main theorem for $n = 3$ and in the appendix.

2. INFINITESIMAL EINSTEIN DEFORMATIONS AND L^2 COHOMOLOGY

2.1. Infinitesimal Einstein deformations on Kähler manifolds. Let us consider the linearized gauged Einstein operator \mathcal{E}'_g of a complete Einstein Kähler metric g with Einstein constant $\lambda < 0$ defined on a complex manifold Ω with dimension n . Thanks to the complex structure, any symmetric 2-tensor $\sigma \in \Gamma(S^2 T^* \Omega)$ decomposes into the sum of the hermitian and the anti-hermitian parts: $\sigma = \sigma_H + \sigma_A$. By definition, these two summands satisfy

$$\sigma_H(J \cdot, J \cdot) = \sigma_H(\cdot, \cdot) \quad \text{and} \quad \sigma_A(J \cdot, J \cdot) = -\sigma_A(\cdot, \cdot),$$

where J denotes the almost complex structure endomorphism. The hermitian (resp. anti-hermitian) part of $S^2 T^* \Omega$ will be denoted by $S^2_H T^* \Omega$ (resp. $S^2_A T^* \Omega$). This decomposition is respected by \mathcal{E}'_g , for the curvature of the Kähler metric has only components of the type $R_{i\bar{j}k\bar{l}}$.

We discuss the action of \mathcal{E}'_g on each component based on Koiso's observation [29, Section 7] (see also Besse [4, Section 12.J]). First, we identify any hermitian symmetric form σ_H with the differential $(1, 1)$ -form $\sigma_H(\cdot, J \cdot)$, which is denoted by $\sigma_H \circ J$. Then the action of \mathcal{E}'_g on the hermitian part is related to that of the Hodge–de Rham Laplacian Δ_d as follows:

$$(\mathcal{E}'_g \sigma_H) \circ J = \frac{1}{2}(\Delta_d - 2\lambda)(\sigma_H \circ J).$$

On complete manifolds, by a result of Gaffney [22] it is known that Δ_d is essentially self-adjoint, meaning that it has unique self-adjoint extension. In particular, the maximal closed extension of Δ_d agrees with $dd^* + d^*d$ (where d also acts distributionally). Therefore, since $\lambda < 0$, it follows that

$$(2.1) \quad \ker_{L^2} \mathcal{E}'_g \cap L^2(S^2_H T^* \Omega) = 0.$$

Second, the action of \mathcal{E}'_g on the anti-hermitian part σ_A is reinterpreted as follows. Let $\sigma_A = \sigma_A^{2,0} + \sigma_A^{0,2}$ be the type decomposition of σ_A , and we identify $\sigma_A^{0,2}$ through the metric duality with a $(0, 1)$ -form with values in $T^{1,0} = T^{1,0} \Omega$, which is denoted by $g^{-1} \circ \sigma_A^{0,2}$. Then

$$(2.2) \quad g^{-1} \circ (\mathcal{E}'_g \sigma_A)^{0,2} = \frac{1}{2} \Delta_{\bar{\partial}} (g^{-1} \circ \sigma_A^{0,2}).$$

By (2.1) and (2.2), we have a natural identification

$$(2.3) \quad \ker_{L^2} \mathcal{E}'_g \cong L^2 \mathcal{H}^{0,1}(T^{1,0}),$$

where $L^2 \mathcal{H}^{0,1}(T^{1,0})$ is the space of L^2 harmonic $T^{1,0}$ -valued $(0, 1)$ -forms:

$$L^2 \mathcal{H}^{0,1}(T^{1,0}) := \{ \alpha \in L^2 \wedge^{0,1}(T^{1,0}) \mid \bar{\partial} \alpha = 0, \bar{\partial}^* \alpha = 0 \} = \{ \alpha \in L^2 \wedge^{0,1}(T^{1,0}) \mid \Delta_{\bar{\partial}} \alpha = 0 \}.$$

The latter equality follows from the essential self-adjointness of $\Delta_{\bar{\partial}}$ due to Chernoff [14].

2.2. Reduction to L^2 cohomology. The Hodge–Kodaira decomposition on noncompact Hermitian manifolds reads as follows, where E is an arbitrary hermitian holomorphic vector bundle:

$$L^2 \wedge^{p,q}(\Omega; E) = L^2 \mathcal{H}^{p,q}(\Omega; E) \oplus \overline{\text{im } \bar{\partial}_{p,q-1}} \oplus \overline{\text{im } \bar{\partial}_{p,q}^*};$$

here

$$\bar{\partial} = \bar{\partial}_{p,q}: L^2 \wedge^{p,q}(\Omega; E) \rightarrow L^2 \wedge^{p,q+1}(\Omega; E)$$

is the maximal closed extension of $\bar{\partial}$ acting on compactly supported smooth (p, q) -forms. Therefore, the space $L^2\mathcal{H}^{p,q}(\Omega; E)$ is isomorphic to the so-called *reduced* L^2 cohomology:

$$L^2\mathcal{H}^{p,q}(\Omega; E) \cong L^2H_{\text{red}}^{p,q}(\Omega; E) := \ker \bar{\partial}_{p,q} / \overline{\text{im } \bar{\partial}_{p,q-1}}.$$

The reduced cohomology can be different from the usual L^2 cohomology

$$L^2H^{p,q}(\Omega; E) := \ker \bar{\partial}_{p,q} / \text{im } \bar{\partial}_{p,q-1}.$$

However, it is clear that $L^2H^{p,q}(\Omega; E) = 0$ implies $L^2H_{\text{red}}^{p,q}(\Omega; E) = 0$. Therefore we can consider the usual L^2 cohomology to get a result on harmonic forms.

Let us recall an exact sequence for L^2 cohomologies. Since an inclusion $K \subset K'$ between compact subsets of Ω induces a homomorphism $L^2H^{p,q}(\Omega \setminus K; E) \rightarrow L^2H^{p,q}(\Omega \setminus K'; E)$ by restriction, we may define the inductive limit

$$\varinjlim_K L^2H^{p,q}(\Omega \setminus K; E),$$

where K runs through the compact subsets of Ω . Then we obtain the following exact sequence (cf. Ohsawa [34]):

$$\cdots \rightarrow H_c^{p,q}(\Omega; E) \rightarrow L^2H^{p,q}(\Omega; E) \rightarrow \varinjlim_K L^2H^{p,q}(\Omega \setminus K; E) \rightarrow H_c^{p,q+1}(\Omega; E) \rightarrow \cdots.$$

Here $H_c^{p,q}(\Omega; E)$ denotes the cohomology with compact support.

Now suppose that Ω is a Stein manifold. Then by a result of Andreotti–Vesentini [2, Theorem 5], $H_c^{0,1}(\Omega; E)$ vanishes for any hermitian vector bundle E . Therefore, by (2.3) and the exact sequence above, the vanishing of $\ker_{L^2} \mathcal{E}'_g$ follows once

$$(2.4) \quad \varinjlim_K L^2H^{0,1}(\Omega \setminus K; T^{1,0}) = 0$$

is shown.

Further consideration is possible if g is an ACH Kähler-Einstein metric. In this case, we apply Proposition 1.4 to show that $\ker_{L^2} \mathcal{E}'_g$ actually lies in the weighted L^2 -space $L^2_\delta(S^2T^*\Omega)$ for $0 < \delta < n$, which implies that

$$(2.5) \quad L^2\mathcal{H}^{0,1}(\Omega; T^{1,0}) \subset L^2_\delta \wedge^{0,1}(\Omega; T^{1,0}).$$

Thus we are led to considering the weighted L^2 cohomology. In this case, the vanishing of $L^2_\delta H^{0,1}(\Omega; T^{1,0})$ follows from

$$(2.6) \quad \varinjlim_K L^2_\delta H^{0,1}(\Omega \setminus K; T^{1,0}) = 0$$

because, if we denote by E_δ the vector bundle $T^{1,0}$ equipped with the metric $\varphi^{-\delta}g$, then the weighted cohomology $L^2_\delta H^{p,q}(\Omega; T^{1,0})$ is nothing but $L^2H^{p,q}(\Omega; E_\delta)$. Now suppose that $L^2_\delta H^{0,1}(\Omega; T^{1,0}) = 0$ is shown, and take any element $\alpha \in L^2\mathcal{H}^{0,1}(\Omega; T^{1,0})$. From (2.5) it follows that $\alpha \in L^2_\delta \wedge^{0,1}(\Omega; T^{1,0})$, and at the same time we have $\bar{\partial}\alpha = 0$. Hence, by the assumption, there is some $\beta \in L^2_\delta(\Omega; T^{1,0})$ for which $\alpha = \bar{\partial}\beta$. Then it turns out that β also belongs to $L^2(\Omega; T^{1,0})$. Now since $\bar{\partial}^*\alpha = 0$, we obtain $\bar{\partial}^*\bar{\partial}\beta = 0$, which implies $\alpha = \bar{\partial}\beta = 0$. Thus we can conclude that $L^2\mathcal{H}^{0,1}(\Omega; T^{1,0})$ vanishes, and so does the obstruction space $\ker_{L^2} \mathcal{E}'_g$.

3. PROOF OF MAIN THEOREM

We shall prove that (2.6) holds for $n \geq 3$; thus our main theorem follows by Proposition 1.3. In the course of the proof of (2.6), we will also see that (2.4) holds when $n \geq 4$. Therefore, the only case one really has to consider the weighted cohomology is when $n = 3$.

Since the L^2 cohomology is invariant for quasi-equivalent metrics, we can replace the Cheng–Yau metric g with the metric \tilde{g} expressed as $\tilde{g}_{i\bar{j}} = \partial_i \partial_{\bar{j}}(-\log \tilde{\varphi})$, where $\tilde{\varphi} \in C^\infty(\bar{\Omega})$ is a *smooth* positive defining function. This simplification avoids annoying differentiability issues. In what follows, we omit tildes: \tilde{g} and $\tilde{\varphi}$ are simply denoted by g and φ , respectively.

3.1. Preliminary considerations. We define $\mathcal{U}_\rho := \{0 < \varphi < \rho\} \subset \Omega$ for small $\rho > 0$ so that $M_\rho = \{\varphi = \rho\}$ is smooth. What we prove in this section is actually the following, which is supposedly stronger than (2.6).

Proposition 3.1. *Let $n \geq 3$. For any positive number $\delta > 0$,*

$$(3.1) \quad L_\delta^2 H^{0,1}(\mathcal{U}_\rho; T^{1,0}) = 0$$

if $\rho > 0$ is sufficiently small.

The claim (3.1) is the solvability of a $\bar{\partial}$ -equation on a complete manifold under the presence of boundary. The proof reduces to establishing the estimate below (see Hörmander [27, Theorem 1.1.4] or [28, Theorem 4.1.1]).

Proposition 3.2. *Let $n \geq 3$ and $\delta > 0$. For sufficiently small $\rho > 0$, there exists a constant $C > 0$ such that*

$$(3.2) \quad \|\alpha\|^2 \leq C(\|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2), \quad \alpha \in \text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}^* \subset L_\delta^2 \wedge^{0,1}(\mathcal{U}_\rho; T^{1,0}).$$

As remarked in the previous section, we may incorporate the weight into the fiber metric of $T^{1,0}$. Therefore we shall present the necessary computation for differential forms on Ω with values in an arbitrary hermitian holomorphic vector bundle E . In addition, we consider differential $(0, q)$ -forms in general, $0 \leq q \leq n$, for it makes the situation clearer.

While the domain of $\bar{\partial} = \bar{\partial}_{0,q}$ contains the space $C_c^\infty \wedge^{0,q}(\bar{\mathcal{U}}_\rho; E)$ of E -valued smooth $(0, q)$ -forms with compact support in $\bar{\mathcal{U}}_\rho = \{0 < \varphi \leq \rho\}$ as a subspace, the domain of $\bar{\partial}^* = \bar{\partial}_{0,q-1}^*$ does not (unless $q = 0$, for which $\bar{\partial}^*$ is trivial). We define

$$\mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E) := C_c^\infty \wedge^{0,q}(\bar{\mathcal{U}}_\rho; E) \cap \text{dom } \bar{\partial}^*.$$

This space is described as follows. Let ξ be the $(1, 0)$ -vector field on $\bar{\mathcal{U}}_\rho$ such that, for each $0 < c \leq \rho$, its restriction $\xi|_{M_c}$ along the level set $M_c = \{\varphi = c\}$ is the unit normal vector field pointing toward $\partial\Omega$. Then $\alpha \in C_c^\infty \wedge^{0,q}(\bar{\mathcal{U}}_\rho; E)$ belongs to $\mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E)$ if and only if

$$(3.3) \quad i\bar{\xi}\alpha = 0 \quad \text{on } M_\rho.$$

The following lemma shows that it suffices to establish the estimates for elements of $\mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E)$.

Lemma 3.3. *The space $\mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E)$ is dense in $\text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}^* \subset L^2 \wedge^{0,q}(\mathcal{U}_\rho; E)$ with respect to the graph norm $\alpha \mapsto (\|\alpha\|^2 + \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2)^{1/2}$.*

Proof. By a partition of unity, we may decompose $\alpha \in \text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}^*$ into the sum $\alpha = \alpha_1 + \alpha_2$, where α_1 is supported near M_ρ and $\text{supp } \alpha_2 \subset \mathcal{U}_\rho$. It suffices to approximate α_1 and α_2 separately by elements of $\mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E)$. Further partition allows us to assume that α_1 is supported in a local boundary chart U of $\bar{\mathcal{U}}_\rho$. Then a result of Hörmander [27, Proposition 1.2.4] (see also Chen–Shaw [12, Lemma 4.3.2]) shows that there exists a sequence $\alpha_1^\nu \in \mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E)$ supported in U

such that $\alpha_1' \rightarrow \alpha_1$ in the graph norm. The second term α_2 is approximated by smooth forms supported in \mathcal{U}_ρ by the standard cut-off technique for complete manifolds (see, e.g., the proof of [2, Lemma 4]). \square

We will later need the divergence of ξ and the commutator $[\xi, \bar{\xi}]$, which can be computed as follows. Recall from Lee–Melrose [31, Section 2] that there exists a unique $(1, 0)$ -vector field X on a (two-sided) neighborhood of $\partial\Omega$ satisfying

$$\iota_X \partial \bar{\partial} \varphi = \kappa \bar{\partial} \varphi, \quad \partial \varphi(X) = -1$$

for some real-valued function κ , which is called the *transverse curvature*. Then, since

$$g = \frac{\partial \varphi \bar{\partial} \varphi}{\varphi^2} - \frac{\partial \bar{\partial} \varphi}{\varphi},$$

we get $|X|^2 = \varphi^{-2}(1 + \kappa\varphi)$ and hence $\xi = (1 + \kappa\varphi)^{-1/2} \varphi X$, which is the metric dual of $(1 + \kappa\varphi)^{1/2} \bar{\partial}(-\log \varphi)$. This implies that

$$\begin{aligned} \text{div } \xi &= \text{tr } \nabla' \xi = \text{tr}_g \partial((1 + \kappa\varphi)^{1/2} \bar{\partial}(-\log \varphi)) \\ (3.4) \quad &= \text{tr}_g \partial \bar{\partial}(-\log \varphi) + o(1) = n + o(1) \quad \text{as } \varphi \rightarrow 0, \end{aligned}$$

where $\nabla = \nabla' + \nabla''$ is the type decomposition of the Levi-Civita connection. Moreover,

$$(3.5) \quad [\xi, \bar{\xi}] = [\varphi X, \varphi \bar{X}] + o(1) = \varphi X - \varphi \bar{X} + \varphi^2 [X, \bar{X}] + o(1) = \xi - \bar{\xi} + o(1) \quad \text{as } \varphi \rightarrow 0,$$

the last equality being because $[X, \bar{X}]$ is continuous up to $\partial\Omega$ and hence has $O(\varphi^{-1})$ pointwise norm with respect to g .

3.2. The estimate. The usual technique for obtaining estimates related to the $\bar{\partial}$ -Neumann problem on strictly pseudoconvex domains is to use the Morrey–Kohn–Hörmander equality, which equates $\|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^* \alpha\|^2$ with $\|\nabla'' \alpha\|^2$ plus zeroth-order terms and a boundary integral. However, in our case, M_ρ is strictly *pseudoconcave* as the boundary of \mathcal{U}_ρ . Hörmander [27] introduced (see also Folland–Kohn [21, Section III.2]) the “condition $Z(q)$ ” to take such cases into consideration. An interpretation of his technique is to use an equality that lies between those of Morrey–Kohn–Hörmander and Bochner–Kodaira–Nakano, the latter being, in this case, a relation between $\|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^* \alpha\|^2$ and $\|\nabla' \alpha\|^2$. We shall apply his approach and write the relevant terms in terms of curvature.

We start with a geometric version of the Morrey–Kohn–Hörmander equality established by Andreotti–Vesentini [2]. Let $\alpha, \beta \in \mathcal{D}^{0,q}(\bar{\mathcal{U}}_\rho; E)$. Using local holomorphic coordinates (z^1, \dots, z^n) and a local holomorphic frame $(s_1, \dots, s_{\text{rank } E})$ of E , we write

$$\alpha = \frac{1}{q!} \alpha_{\bar{j}_1 \dots \bar{j}_q}^a d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} \otimes s_a,$$

where the sum is taken over all $(j_1, \dots, j_q) \in \{1, \dots, n\}^q$ (not only over the increasing indices) and $a \in \{1, \dots, \text{rank } E\}$, and $\alpha_{\bar{j}_1 \dots \bar{j}_q}^a$ is skew-symmetric in j_1, \dots, j_q . Then we define

$$\langle \alpha, \beta \rangle := \frac{1}{q!} \alpha_{\bar{j}_1 \dots \bar{j}_q}^a \bar{\beta}^{\bar{j}_1 \dots \bar{j}_q}_a \quad \text{and} \quad (\alpha, \beta) := \int_{\mathcal{U}_\rho} \langle \alpha, \beta \rangle dV_g.$$

The latter can be explicitly written as $(\alpha, \beta)_{L^2(\mathcal{U}_\rho)}$, but we suppress $L^2(\mathcal{U}_\rho)$ for notational simplicity. The L^2 -norm of α on \mathcal{U}_ρ is defined by $\|\alpha\| = (\alpha, \alpha)^{1/2}$. Moreover, we write $|\alpha|^2 = \langle \alpha, \alpha \rangle$ and

$$\|\alpha\|_b^2 := \int_{M_\rho} |\alpha|^2 dS_g,$$

where dS_g is the area measure on M_ρ induced by dV_g . The actions of the Ricci tensor of g and the curvature $S = S_{i\bar{j}}^a{}^b$ of E are defined as follows, where the square bracket notation means that we take the skew-symmetrization over the indices $\bar{j}_1, \bar{j}_2, \dots, \bar{j}_q$:

$$\begin{aligned} (\text{Ric}^\circ \alpha)_{\bar{j}_1 \dots \bar{j}_q}^a &:= \sum_{s=1}^q \text{Ric}_{\bar{j}_s}^{\bar{k}} \alpha_{\bar{j}_1 \dots \bar{k} \dots \bar{j}_q}^a = q \text{Ric}_{[\bar{j}_1]}^{\bar{k}} \alpha_{\bar{k}|\bar{j}_2 \dots \bar{j}_q]}^a, \\ (\mathring{S}\alpha)_{\bar{j}_1 \dots \bar{j}_q}^a &:= \sum_{s=1}^q S_{\bar{j}_s b}^{\bar{k}} \alpha_{\bar{j}_1 \dots \bar{k} \dots \bar{j}_q}^b = q S_{[\bar{j}_1]}^{\bar{k}} \alpha_{\bar{k}|\bar{j}_2 \dots \bar{j}_q]}^b. \end{aligned}$$

Then, using (3.3), we get (see [2, p. 113])

$$\|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 = \|\nabla''\alpha\|^2 + (\text{Ric}^\circ \alpha, \alpha) + (\mathring{S}\alpha, \alpha) - q \int_{\partial\mathcal{U}_\rho} \frac{1}{|\partial \log \varphi|} |\alpha|^2.$$

The asymptotic curvature behavior (1.2) implies

$$\text{Ric}^\circ \alpha = -q(n+1)\alpha + o(1).$$

Moreover, $|\partial \log \varphi| = (1 + \kappa\varphi)^{1/2}$ and thus it tends to 1 uniformly at $\partial\Omega$. Therefore,

$$(3.6) \quad \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 = \|\nabla''\alpha\|^2 - q(n+1)\|\alpha\|^2 + (\mathring{S}\alpha, \alpha) - q\|\alpha\|_b^2 + o(\|\alpha\|^2 + \|\alpha\|_b^2),$$

where the remainder term being $o(\|\alpha\|^2 + \|\alpha\|_b^2)$ means that, for any $\varepsilon > 0$, if $\rho > 0$ is sufficiently small then the absolute value of this term is bounded by $\varepsilon(\|\alpha\|^2 + \|\alpha\|_b^2)$.

Andreotti–Vesentini equality (3.6) has two defects for our purpose: the presence of a negative boundary integral $-q\|\alpha\|_b^2$ and the fact that the curvature term becomes negative. Both can be remedied by decomposing $\|\nabla''\alpha\|^2$ into the tangential and normal parts,

$$\|\nabla''\alpha\|^2 = \|\nabla_b''\alpha\|^2 + \|\nabla_{\bar{\xi}}\alpha\|^2,$$

and replacing $\|\nabla_b''\alpha\|^2$ with $\|\nabla_b'\alpha\|^2$ by integration-by-parts.

Lemma 3.4. *For $\alpha \in C_c^\infty \wedge^{0,q}(\bar{\mathcal{U}}_\rho; E)$,*

$$\begin{aligned} \|\nabla_b''\alpha\|^2 &= \|\nabla_b'\alpha\|^2 + n(n+q-1)\|\alpha\|^2 - \|\iota_{\bar{\xi}}\alpha\|^2 - ((\text{tr}_g S)\alpha, \alpha) + (S(\xi, \bar{\xi})\alpha, \alpha) \\ &\quad + 2(n-1) \text{Re}(\nabla_{\bar{\xi}}\alpha, \alpha) + (n-1)\|\alpha\|_b^2 + o(\|\alpha\|^2 + \|\nabla_b\alpha\|^2 + \|\nabla_{\bar{\xi}}\alpha\|^2 + \|\alpha\|_b^2). \end{aligned}$$

Proof. We first compute the difference between $\|\nabla''\alpha\|^2$ and $\|\nabla'\alpha\|^2$. By the divergence theorem,

$$\|\nabla''\alpha\|^2 = -(\text{tr}_g \nabla' \nabla'' \alpha, \alpha) - \int_{M_\rho} \langle \nabla_{\bar{\xi}} \alpha, \alpha \rangle,$$

where the trace of $\nabla' \nabla'' \alpha \in \Omega^{1,0} \otimes \Omega^{0,1} \otimes \Omega^{0,q}$ is taken in the first $\Omega^{1,0} \otimes \Omega^{0,1}$ factors. The trace can be rewritten as

$$\text{tr}_g \nabla' \nabla'' \alpha = \text{tr}_g \nabla'' \nabla' \alpha + \text{Ric}^\circ \alpha + (\text{tr}_g S)\alpha.$$

Hence

$$\begin{aligned} \|\nabla''\alpha\|^2 &= -(\text{tr}_g \nabla'' \nabla' \alpha, \alpha) - (\text{Ric}^\circ \alpha, \alpha) - ((\text{tr}_g S)\alpha, \alpha) - \int_{M_\rho} \langle \nabla_{\bar{\xi}} \alpha, \alpha \rangle \\ (3.7) \quad &= \|\nabla'\alpha\|^2 - (\text{Ric}^\circ \alpha, \alpha) - ((\text{tr}_g S)\alpha, \alpha) + \int_{M_\rho} \langle \nabla_{\xi-\bar{\xi}} \alpha, \alpha \rangle \\ &= \|\nabla'\alpha\|^2 + q(n+1)\|\alpha\|^2 - ((\text{tr}_g S)\alpha, \alpha) + \int_{M_\rho} \langle \nabla_{\xi-\bar{\xi}} \alpha, \alpha \rangle + o(\|\alpha\|^2). \end{aligned}$$

Next we compute the difference between $\|\nabla_{\bar{\xi}}\alpha\|^2$ and $\|\nabla_{\xi}\alpha\|^2$. Again by the divergence theorem,

$$\begin{aligned}\|\nabla_{\bar{\xi}}\alpha\|^2 &= -(\nabla_{\xi}\nabla_{\bar{\xi}}\alpha, \alpha) - ((\operatorname{div} \xi)\nabla_{\bar{\xi}}\alpha, \alpha) - \int_{M_{\rho}} \langle \nabla_{\bar{\xi}}\alpha, \alpha \rangle \\ &= -(\nabla_{\bar{\xi}}\nabla_{\xi}\alpha + R(\xi, \bar{\xi})\alpha + S(\xi, \bar{\xi})\alpha + \nabla_{[\xi, \bar{\xi}]} \alpha, \alpha) - ((\operatorname{div} \xi)\nabla_{\bar{\xi}}\alpha, \alpha) - \int_{M_{\rho}} \langle \nabla_{\bar{\xi}}\alpha, \alpha \rangle.\end{aligned}$$

Since $R(\xi, \bar{\xi})\alpha = -q\alpha - q\xi_b \wedge \iota_{\xi}\alpha + o(|\alpha|)$ by (1.2), where ξ_b is the metric dual of ξ , we get

$$(R(\xi, \bar{\xi})\alpha, \alpha) = -q\|\alpha\|^2 - \|\iota_{\xi}\alpha\|^2 + o(\|\alpha\|^2).$$

Moreover,

$$\|\nabla_{\xi}\alpha\|^2 = -(\nabla_{\bar{\xi}}\nabla_{\xi}\alpha, \alpha) - ((\operatorname{div} \bar{\xi})\nabla_{\xi}\alpha, \alpha) - \int_{M_{\rho}} \langle \nabla_{\xi}\alpha, \alpha \rangle.$$

As a result, we can conclude that

$$\begin{aligned}\|\nabla_{\bar{\xi}}\alpha\|^2 &= \|\nabla_{\xi}\alpha\|^2 + q\|\alpha\|^2 + \|\iota_{\bar{\xi}}\alpha\|^2 - (S(\xi, \bar{\xi})\alpha, \alpha) - (\nabla_{[\xi, \bar{\xi}]} \alpha, \alpha) \\ &\quad - ((\operatorname{div} \xi)\nabla_{\bar{\xi}}\alpha, \alpha) + ((\operatorname{div} \bar{\xi})\nabla_{\xi}\alpha, \alpha) + \int_{M_{\rho}} \langle \nabla_{\xi-\bar{\xi}}\alpha, \alpha \rangle + o(\|\alpha\|^2).\end{aligned}$$

By (3.4) and (3.5),

$$\begin{aligned}&((\operatorname{div} \bar{\xi})\nabla_{\xi}\alpha, \alpha) - ((\operatorname{div} \xi)\nabla_{\bar{\xi}}\alpha, \alpha) - (\nabla_{[\xi, \bar{\xi}]} \alpha, \alpha) \\ &= (n-1)(\nabla_{\xi-\bar{\xi}}\alpha, \alpha) + (\nabla_{f\xi-\bar{f}\bar{\xi}}\alpha, \alpha) + o(\|\alpha\|^2 + \|\nabla_b\alpha\|^2),\end{aligned}$$

where f is a smooth function defined near $\partial\Omega$ that vanishes along $\partial\Omega$. Summarizing, we have

$$\begin{aligned}\|\nabla_{\bar{\xi}}\alpha\|^2 &= \|\nabla_{\xi}\alpha\|^2 + q\|\alpha\|^2 + \|\iota_{\bar{\xi}}\alpha\|^2 - (S(\xi, \bar{\xi})\alpha, \alpha) + (n-1)(\nabla_{\xi-\bar{\xi}}\alpha, \alpha) \\ &\quad + (\nabla_{f\xi-\bar{f}\bar{\xi}}\alpha, \alpha) + \int_{M_{\rho}} \langle \nabla_{\xi-\bar{\xi}}\alpha, \alpha \rangle + o(\|\alpha\|^2 + \|\nabla_b\alpha\|^2).\end{aligned}$$

Now

$$\begin{aligned}(\nabla_{\xi}\alpha, \alpha) &= \int_{\mathcal{U}_{\rho}} \xi|\alpha|^2 - \overline{(\nabla_{\bar{\xi}}\alpha, \alpha)} = - \int_{\mathcal{U}_{\rho}} (\operatorname{div} \xi)|\alpha|^2 - \|\alpha\|_b^2 - \overline{(\nabla_{\bar{\xi}}\alpha, \alpha)} \\ &= -n\|\alpha\|^2 - \|\alpha\|_b^2 - \overline{(\nabla_{\bar{\xi}}\alpha, \alpha)} + o(\|\alpha\|^2)\end{aligned}$$

and similarly one gets $(\nabla_{f\xi}\alpha, \alpha) = o(\|\alpha\|^2 + \|\nabla_{\bar{\xi}}\alpha\|^2 + \|\alpha\|_b^2)$. Therefore,

$$\begin{aligned}\|\nabla_{\bar{\xi}}\alpha\|^2 &= \|\nabla_{\xi}\alpha\|^2 + q\|\alpha\|^2 + \|\iota_{\bar{\xi}}\alpha\|^2 - (S(\xi, \bar{\xi})\alpha, \alpha) - n(n-1)\|\alpha\|^2 - 2(n-1)\operatorname{Re}(\nabla_{\bar{\xi}}\alpha, \alpha) \\ &\quad - (n-1)\|\alpha\|_b^2 + \int_{M_{\rho}} \langle \nabla_{\xi-\bar{\xi}}\alpha, \alpha \rangle + o(\|\alpha\|^2 + \|\nabla_b\alpha\|^2 + \|\nabla_{\bar{\xi}}\alpha\|^2 + \|\alpha\|_b^2).\end{aligned}$$

Combining this with (3.7), we obtain the lemma. \square

Equation (3.6) and Lemma 3.4 imply the following approximate equality for $\alpha \in \mathcal{D}^{0,q}(\bar{\mathcal{U}}_{\rho}; E)$:

$$\begin{aligned}(3.8) \quad \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 &= \|\nabla'_b\alpha\|^2 + \|\nabla_{\bar{\xi}}\alpha\|^2 + (n^2 - n - q)\|\alpha\|^2 - \|\iota_{\bar{\xi}}\alpha\|^2 \\ &\quad + (\bar{S}\alpha - (\operatorname{tr}_g S)\alpha + S(\xi, \bar{\xi})\alpha, \alpha) - 2(n-1)\operatorname{Re}(\nabla_{\bar{\xi}}\alpha, \alpha) + (n-q-1)\|\alpha\|_b^2 \\ &\quad + o(\|\alpha\|^2 + \|\nabla'_b\alpha\|^2 + \|\nabla_{\bar{\xi}}\alpha\|^2 + \|\alpha\|_b^2).\end{aligned}$$

As a consequence, we obtain the following estimate.

Proposition 3.5. *For $\alpha \in \mathcal{D}^{0,q}(\overline{\mathcal{U}}_\rho; E)$*

$$(3.9) \quad \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 \geq (\mathring{S}\alpha - (\text{tr}_g S)\alpha + S(\xi, \bar{\xi})\alpha, \alpha) + (n - q - 2)\|\alpha\|^2 + (n - q - 1)\|\alpha\|_b^2 + o(\|\alpha\|^2 + \|\alpha\|_b^2)$$

in the sense that, for any given $\varepsilon > 0$, if $\rho > 0$ is sufficiently small then inequality (3.9) with $o(\|\alpha\|^2 + \|\alpha\|_b^2)$ replaced by $-\varepsilon(\|\alpha\|^2 + \|\alpha\|_b^2)$ holds for any $\alpha \in \mathcal{D}^{0,q}(\overline{\mathcal{U}}_\rho; E)$.

Proof. By the Cauchy–Schwarz inequality, we have

$$(3.10) \quad (1 - \varepsilon')\|\nabla \bar{\xi}\alpha\|^2 - 2(n - 1)\text{Re}(\nabla \bar{\xi}\alpha, \alpha) \geq -(1 - \varepsilon')^{-1}(n - 1)^2\|\alpha\|^2.$$

The proposition follows from (3.8), (3.10), and $\|\iota \bar{\xi}\alpha\|^2 \leq \|\alpha\|^2$. \square

We apply this proposition to $E = E_\delta = (T^{1,0}, \varphi^{-\delta}g)$. Then, since

$$S = R - \delta \bar{\partial} \bar{\partial}(-\log \varphi) \otimes I = R - \delta g \otimes I,$$

we get

$$\begin{aligned} (\mathring{S}\alpha, \alpha) &\geq -(2q + q\delta)\|\alpha\|^2 + o(\|\alpha\|^2), \\ ((\text{tr}_g S)\alpha, \alpha) &= -(n + 1 + n\delta)\|\alpha\|^2 + o(\|\alpha\|^2), \\ (S(\xi, \bar{\xi})\alpha, \alpha) &\geq -(2 + \delta)\|\alpha\|^2 + o(\|\alpha\|^2). \end{aligned}$$

Therefore, for any $\varepsilon > 0$, if $\rho > 0$ is small enough then

$$(3.11) \quad \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 \geq (2n - 3q - 3 - \varepsilon)\|\alpha\|^2 + (n - q - 1)\delta\|\alpha\|^2 + (n - q - 1 - \varepsilon)\|\alpha\|_b^2$$

for any $\alpha \in \mathcal{D}^{0,q}(\overline{\mathcal{U}}_\rho; E)$. Thus Proposition 3.2 follows by Lemma 3.3, and the proof of our main theorem is completed. It is also obvious from (3.11) that, if $n \geq 4$, then we actually do not need the weight $\varphi^{-\delta}$.

APPENDIX A. ON THE VANISHING RESULT OF DONNELLY AND FEFFERMAN

Recall the following theorem on the space $L^2\mathcal{H}^{p,q}(\Omega)$ of L^2 harmonic (p, q) -forms due to Donnelly and Fefferman (which is restated in a way that is convenient for us).

Theorem A.1 (Donnelly–Fefferman [16], Donnelly [15]). *Let Ω be a smoothly bounded strictly pseudoconvex domain of a Stein manifold Y equipped with the Cheng–Yau metric. Then,*

$$(A.1) \quad \dim L^2\mathcal{H}^{p,q}(\Omega) = \begin{cases} 0, & p + q \neq n, \\ \infty, & p + q = n. \end{cases}$$

Actually, Donnelly–Fefferman [16] considered the case in which $Y = \mathbb{C}^n$, and the metric was the Bergman metric (which is in fact quasi-equivalent to the Cheng–Yau metric). For this case, Berndtsson [3] has given another proof for $(p, q) = (n, 1)$ in connection with the extension theorem of Ohsawa–Takegoshi. The result of Donnelly [15] is more far-reaching: it applies to any complex manifold Ω equipped with a complete Kähler metric g whose associated 2-form ω admits the expression $\omega = d\eta$ with a 1-form η bounded with respect to g . It is based on an observation of Gromov [26].

We shall see in this appendix that our technique can also be used to give another proof of the vanishing part of Theorem A.1. As we did in Section 3, instead of the Cheng–Yau metric, we can consider a metric g of the form

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}}(-\log \varphi),$$

where φ is some smooth defining function of Ω . We apply the theory of geometric elliptic differential operators outlined in Subsection 1.3 to the Dolbeault Laplacian $\Delta_{\bar{\partial}}$.

By the Poincaré duality, it suffices to show $L^2\mathcal{H}^{p,q}(\Omega) = 0$ for $p + q < n$. For these cases, on the complex hyperbolic space $\mathbb{C}H^n$, the coercivity estimate

$$\|\alpha\|^2 \leq C\|\Delta_{\bar{\partial}}\alpha\|^2, \quad \alpha \in \text{dom } \Delta_{\bar{\partial}} \subset L^2\wedge^{p,q}(\Omega)$$

follows by the argument in [16, Section 3] based on a formula of Donnelly–Xavier [17]. This makes Proposition 1.4 applicable. Our claim is the following.

Lemma A.2. *If $p + q < n$, then the indicial radius $R_{\Delta_{\bar{\partial}}}$ of the Dolbeault Laplacian $\Delta_{\bar{\partial}}$ acting on (p, q) -forms is positive. Therefore, by Proposition 1.4, the space $L^2\mathcal{H}^{p,q}(\Omega)$ is contained in $L^2_{\delta}\mathcal{H}^{p,q}(\Omega)$ for some $\delta > 0$.*

This lemma reduces the vanishing of $L^2\mathcal{H}^{p,q}(\Omega)$ to that of the weighted cohomology

$$(A.2) \quad L^2_{\delta}\mathcal{H}^{p,q}(\Omega) = 0$$

by the same argument as in the last paragraph of Subsection 2.2. The point is that (A.2) can be shown by the Bochner–Kodaira–Nakano equality in the usual way. Let L_{δ} be the trivial line bundle equipped with the fiber metric $\varphi^{-\delta}$. Then the curvature S of L_{δ} is given by

$$S_{i\bar{j}} = -\delta\partial_i\bar{\partial}_{\bar{j}}(-\log\varphi) = -\delta g_{i\bar{j}}.$$

Therefore, for compactly supported smooth (p, q) -form $\alpha \in C_c^{\infty}\wedge^{p,q}(\Omega; L_{\delta})$ with values in L_{δ} , one has

$$\|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 = \|D'\alpha\|^2 + \|(D')^*\alpha\|^2 + (n - p - q)\delta\|\alpha\|^2,$$

where D' is the holomorphic part of the covariant exterior derivative. Since $C_c^{\infty}\wedge^{p,q}(\Omega; L_{\delta})$ is dense in $\text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}^*$, this implies (A.2).

The computation of $R_{\Delta_{\bar{\partial}}}$ is tedious but straightforward, no matter which of the methods of [5] and [32] is used. The bundle of (p, q) -forms is associated to the representation $\bigwedge^p \mathfrak{m}_0^* \otimes \bigwedge^q \overline{\mathfrak{m}}_0^*$ in the notation of Subsection 1.3 (where \otimes denotes the tensor product over \mathbb{C}). On the subspace $\bigwedge^p(\mathbb{C}^{n-1})^* \otimes_0 \bigwedge^q(\mathbb{C}^{n-1})^*$, where \otimes_0 means that we take the totally trace-free part, the indicial roots $p + q$ and $2n - p - q$ appear. These are the closest roots to the line $\text{Re } s = n$, which means that $R_{\Delta_{\bar{\partial}}} = n - p - q$. The verification is left to the interested reader.

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